J. Fluid Mech. (1975), *vol.* 72, *part* 4, *pp.* 649–660
Printed in Great Britain

The large amplitude motion of a liquid-filled gyroscope and the non-interaction of inertial and Rossby waves

By W. E. SCOTT

U.S. Army Ballistic Research Laboratories and Department of Earth and Planetary Sciences, The Johns Hopkins University, Baltimore, Maryland 21218

(Received 15 March 1974 and in revised form 21 June 1974)

An attempt is made to explain theoretically two curious phenomena involving the motion of the liquid in a spinning, gyrating, liquid-filled gyroscope. One of the phenomena is the periodic breakdown of the free-surface wave form of the spinning liquid in the gyroscope when it gyrates at angles larger than about 1°. The other is the resonant amplitude growth rate of the liquid-filled gyroscope at these angles, for then the small angle stability theory of Stewartson (1959) fails to make the correct predictions.

The analysis exploits the experimental fact that the axis of rotation of liquid in the rotor of a spinning gyrating gyroscope does not remain coincident with the axis of rotation of the rotor when the gyroscope gyrates at amplitudes greater than the above-mentioned 1°. It is shown that this lack of coincidence generates Rossby waves and modifies the inertial wave frequencies that would ordinarily occur in a right circular cylinder. There is no nonlinear interaction between these Rossby and inertial waves; hence the free-surface breakdown remains unexplained. However, the modification of the inertial wave frequencies does seem to account for the curious amplitude growth rate.

1. Introduction

Stewartson (1959) remarked that his liquid-filled top could be made unstable at almost any filling ratio by making the motion 'large angle', i.e. by causing the axis of the top to depart significantly from the vertical. While verifying a similar phenomenon with a gyroscope partially filled with liquid, we observed two additional interesting phenomena associated with the 'large angle' motion. One was the failure of the axis of the hollow cylindrical core of the liquid to remain coincident with the axis of rotation of the right circular cylindrical rotor as the latter gyrated through large angles. The other phenomenon was the periodic appearance and demise, but only at large angles, of the free-surface inertial wave form shown in figure 1. Since the angular motion of a projectile about its centre of mass is like that of a gyroscope, these phenomena may have some relevance to the stability of spin-stabilized liquid-payload projectiles, for the stability of these projectiles at large angles does not conform to the predictions of the only available theory, the Stewartson small angle stability theory. Figures 2 and 3 show how, for a gyroscope, the amplitude amplification rate at small angles differs from that at large angles.

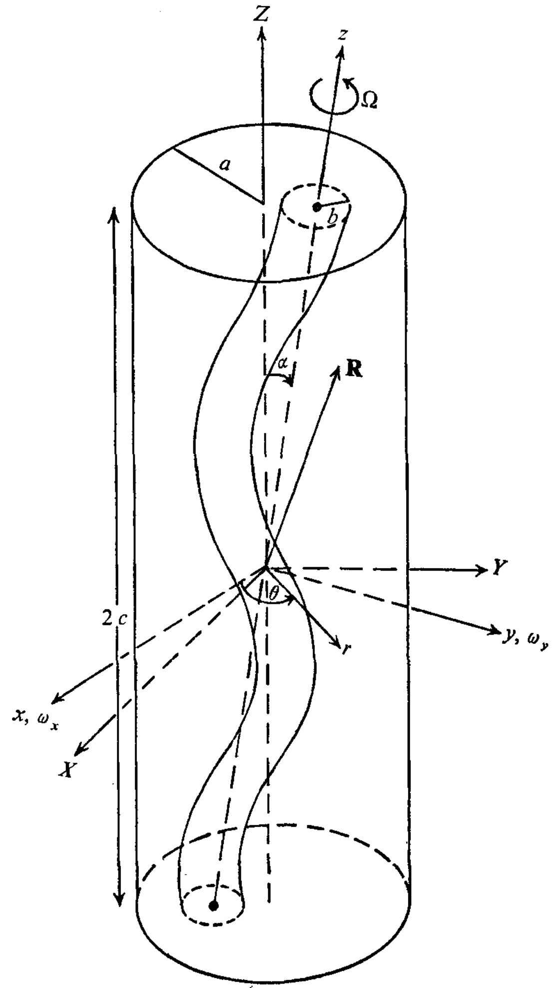

FIGURE 1. Partially filled, spinning cylinder gyrating through a large angle α.

Since 'beat' waves may arise from the interaction of two different inertial wave modes, we initially conjectured that the free-surface breakdown was a manifestation of such beats. However, no combination of the several resonant inertial modes gave a frequency close to that of the periodic breakdown. Furthermore, the fact that the free-surface wave form reappeared not only dismissed turbulence as a factor, but also implied that the phenomenon was reversible, a property not characteristic of instabilities. Hence we conjectured that the periodic breakdown, i.e. the periodic degeneration of the inertial wave form, restoring an axisymmetric surface, was associated with the presence of some other kind of wave in the gyrating rotor. Recalling that Pedlosky & Greenspan (1967) showed that a uniformly rotating liquid in a sliced-off cylinder could support Rossby waves in addition to the ordinary inertial waves, we reasoned that Rossby waves should also arise in the gyroscope at large angles, for the above-mentioned non-coincidence of the two spin vectors in effect should cause the liquid to 'see'

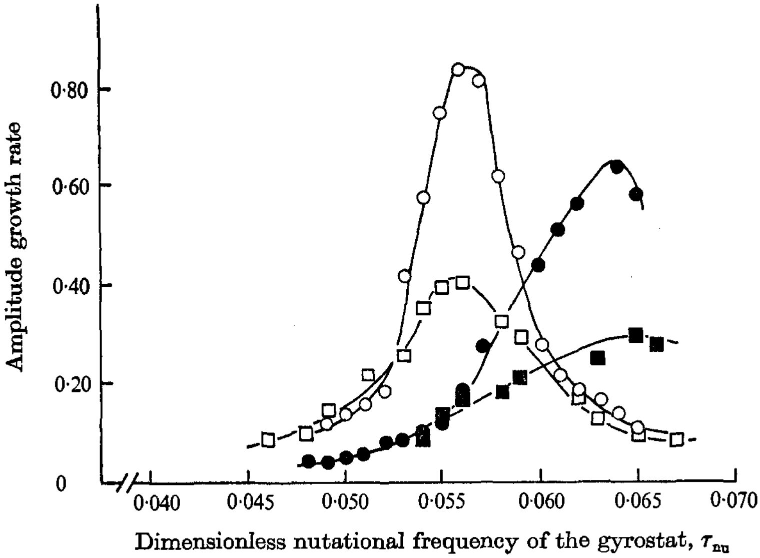

FIGURE 2. Amplitude growth rate *vs.* gyroscopic frequency for a 77 % filled gyroscope. $2c = 7{\cdot}48$ in., $2a = 2{\cdot}50$ in., $\Omega = 5000$ r.p.m. Small amplitude growth rates: ○, 1 cS oil; □, 13 cS oil. Large amplitude growth rates: ●, 1 cS oil; ■, 13 cS oil.

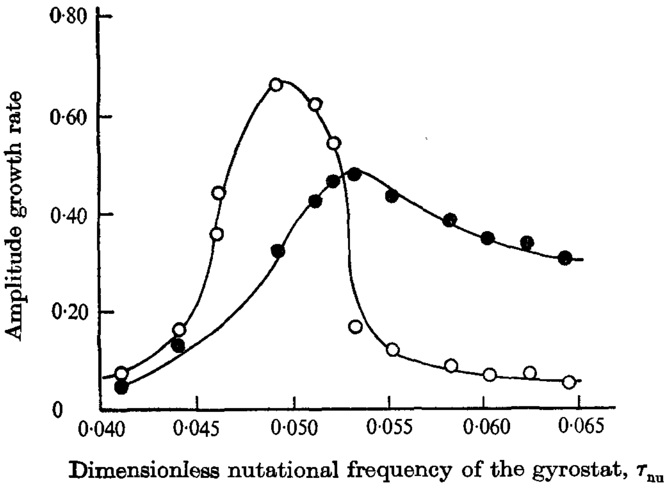

FIGURE 3. Amplitude growth rate *vs.* gyroscopic frequency for a completely filled gyroscope. $2c = 7{\cdot}817$ in., $2a = 2{\cdot}48$ in., $\Omega = 5000$ r.p.m., 1 cS oil. ○, small amplitude growth rate; ●, large amplitude growth rate.

a sliced-off cylinder. In addition, the ordinary inertial waves should still appear, although in some modified form. Hence, in the analysis, we searched for Rossby modes and modified inertial modes with the hope that their interaction, or merely their presence, would account not only for the periodic breakdown, but also the curious amplitude growth rate. The latter phenomenon would be explicable if the inertial wave frequencies increased with the amplitude of the motion, thus requiring the gyroscope to have a higher frequency in order to effect resonance at larger angles, as figures 2 and 3 imply.

The serious implications of figures 2 and 3 should be stressed: a spinning liquid-carrying vehicle designed to operate far from resonance by use of the Stewartson theory may yet experience instability if the amplitude of its motion exceeds a degree or so.

2. Analysis

We follow rather closely Stewartson's (1959) analysis of a liquid-filled spinning top, Pedlosky & Greenspan's (1967) analysis of a sliced-off cylinder and Greenspan's (1969) analysis of nonlinear interaction of inertial modes.

In figure 1 we show a rotating, gyrating, partially filled, right circular cylinder whose axis makes a 'large' angle α with the vertical. Shown also are representative inertial wave forms of two 'opposite' parts of the free surface of the non-aligned hollow central core, whose axis departs from the axis of the cylinder by the angle α. At the moment of breakdown, the free surface becomes essentially axisymmetric.

The equations

To make the analysis tractable, we assume that the angle α remains constant (in experiments it does not!). As in figure 1, we take the X, Y and Z axes to be fixed relative to the cylinder, which has angular velocity components ω_X, ω_Y and ω_Z in the X, Y and Z directions respectively, and the x, y and z axes to be fixed relative to the 'bulk' motion of the fluid, which has angular velocity $\boldsymbol{\omega}$ with components ω_x, ω_y and Ω in the x, y and z directions respectively.

When the gyroscope is gyrating, the Euler equation is

$$\frac{\partial \mathbf{q}}{\partial t}+2\boldsymbol{\omega}\times\mathbf{q} = -\nabla\left(\frac{p'}{\rho}+\tfrac{1}{2}\mathbf{q}.\mathbf{q}\right)+\mathbf{R}\times\frac{\partial \boldsymbol{\omega}}{\partial t}+\mathbf{q}\times(\nabla\times\mathbf{q}), \qquad (1)$$

where gravity has been neglected, $\mathbf{q}$, which satisfies $\nabla.\mathbf{q} = 0$ in the cavity and $\mathbf{q}.\mathbf{n} = 0$ (with $\mathbf{n}$ the normal to the surface in the X, Y, Z frame) at a boundary, is the fluid velocity with respect to the x, y, z frame and

$$p'/\rho = p/\rho+\tfrac{1}{2}\Omega^2 b^2-\tfrac{1}{2}(\boldsymbol{\omega}\times\mathbf{R}).(\boldsymbol{\omega}\times\mathbf{R}).$$

Since we are concerned with the appearance of very low frequency (in the rotating x, y, z frame) inertial waves, i.e. Rossby waves, all of which will disappear when α is zero, we invoke the method of multiple time scales and seek solutions of the form

$$\mathbf{q}(\mathbf{R},t) = \mathbf{Q}(\mathbf{R},t)+\alpha\mathbf{Q}_0(\mathbf{R},\tau,T,\psi,\ldots)+\alpha^2\mathbf{Q}_1(\mathbf{R},\tau,T,\psi,\ldots)+\ldots, \qquad (2)$$

$$p'(R,t) = P(\mathbf{R},t)+\alpha P_0(\mathbf{R},\tau,T,\psi,\ \ldots)+\alpha^2 P_1(\mathbf{R},\tau,T,\psi,\ldots)+\ldots, \qquad (3)$$

where $$\tau = \alpha t, \quad T = \alpha^2 t, \quad \psi = \alpha^3 t, \ldots,$$

$$\mathbf{Q}(R,t) = \sum_m A_m(t)\,\mathbf{Q}_m(\mathbf{R}), \quad P(R,t) = \sum_m A_m(t)\,P_m(R),$$

$\mathbf{Q}_m(R)$ being an inertial mode which satisfies

$$i\lambda\Omega\mathbf{Q}_m(\mathbf{R})+2\Omega\mathbf{k}\times\mathbf{Q}_m(\mathbf{R}) = -\nabla P_m(\mathbf{R}) \qquad (4)$$

and $$\mathbf{Q}_m(\mathbf{R}).\mathbf{n} = 0 \qquad (5)$$

on the wetted surface of the cavity. ($\mathbf{k}$ is a unit vector in the z direction.) We are, of course, assuming that the velocity may be represented by a sum of inertial and Rossby modes, the $\mathbf{Q}_i$ being the Rossby modes.

Now, from (4) we have

$$\mathbf{Q}_m(\mathbf{R}) = \frac{i\lambda_m}{4-\lambda_m^2}\left[\frac{-2\mathbf{k}}{i\lambda_m}\times\left(\frac{2}{i\lambda_m}\right)^2\mathbf{kk}.\right]\nabla P_m(\mathbf{R}), \tag{6}$$

where

$$P_m(R) = \{AJ_1(dr[(2/\lambda_m)^2-1]^{\frac{1}{2}})+BY_1(dr[(2/\lambda_m^2)-1]^{\frac{1}{2}})\}e^{i\theta}\cos[d(z+c)] \tag{7}$$

and satisfies $\nabla^2 P_m+(2/i\lambda_m)^2\,\partial^2 P_m/\partial z^2 = 0$. In the expression for P_m,

$$d = (2j+1)\,\pi/2c, \quad j = 0, 1, \ldots,$$

$2c$ is the cavity height, A and B are constants, and λ_m (for a cylindrical annulus of liquid of inner and outer radii a and b respectively) is determined from

$$\begin{aligned}&[d\beta J_1'(d\beta a)+(2/a\lambda_m)\,J_1(ad\beta)]\,\{d\beta Y_1'(bd\beta)+[2/\lambda^m b+(4-\lambda_m^2)/b]\,Y_1(bd\beta)\}\\ &\quad-[d\beta Y_1'(ad\beta)+(2/a\lambda_m)\,Y_1(ad\beta)]\,\{d\beta J_1'(bd\beta)+[2/\lambda_m b+(4-\lambda_m^2)/b]\,J_1(bd\beta)\} = 0,\end{aligned} \tag{8}$$

where $\beta = [(2/\lambda_m)^2-1]^{\frac{1}{2}}$ and r is a cylindrical co-ordinate. According to (8), λ_m depends critically upon the value of $c/a(2j+1) = \pi/2ad$, where $2j+1$ is the number of half cosine wave forms that can be 'fitted' into the height $2c$ of the cylinder. Hence, since this equation assumes that the liquid is spinning about the central axis of the cylinder, i.e. $\alpha = 0$, we attempt to allow for the effect of a non-zero value of α on the number of waves by setting $d = d_0+\alpha d_1+\alpha^2 d_2+\ldots$, where $d_0 = (2j+1)\,\pi/2c$. Hence

$$\mathbf{Q}_m(\mathbf{R}) = \mathbf{Q}_m(\mathbf{R}, d_0+\alpha d_1+\ldots) = \mathbf{Q}_{m0}(\mathbf{R}, d_0)+\alpha d_1\,\partial\mathbf{Q}_{m0}(\mathbf{R}, d_0)/\partial d_0+\ldots$$

and

$$P_m(\mathbf{R}) = P_{m0}(\mathbf{R}, d_0)+\alpha d_1\,\partial P_{m0}(\mathbf{R}, d_0)/\partial d_0+\ldots.$$

We non-dimensionalize our equations by setting

$$t = t'/\Omega, \quad \nabla = \nabla'/c, \quad \rho = \rho'/c^3, \quad \boldsymbol{\omega} = \alpha\Omega\boldsymbol{\omega}', \quad \mathbf{R} = c\mathbf{R}', \quad P = \alpha\Omega^2\rho'/c$$

and then dropping primes. Setting $\boldsymbol{\omega} = \mathbf{w}\,e^{i\gamma t}$ and

$$A_m(t) = A_{m0}(t,\tau,T,\ldots)+\alpha A_{m1}(t,\tau,T,\ldots)+\alpha^2 A_{m2}(t,\tau,T,\ldots)+\ldots,$$

we have from (1) and (4)

$$\begin{aligned}&\sum_m\left(\frac{\partial A_{m0}}{\partial t}+\alpha\frac{\partial A_{m0}}{\partial\tau}+\alpha^2\frac{\partial A_{m0}}{\partial T}+\ldots+\alpha\frac{\partial A_{m1}}{\partial t}+\alpha^2\frac{\partial A_{m1}}{\partial\tau}+\ldots+\alpha^2\frac{\partial A_{m2}}{\partial t}\right.\\ &\quad\left.+\ldots-i\lambda_{m0}A_{m0}-i\alpha\lambda_{m0}A_{m1}-i\alpha^2\lambda_{m0}A_{m2}-\ldots\right)\left(1+\alpha d_1\frac{\partial}{\partial d_0}+\ldots\right)\mathbf{Q}_{m0}+\alpha^2\frac{\partial\mathbf{Q}_0}{\partial\tau}\\ &\quad+2\alpha\mathbf{k}\times\mathbf{Q}_0+2\alpha^2\mathbf{k}\times\mathbf{Q}_1 = -\nabla(\alpha P_0+\alpha^2 P_1+\ldots)+i\gamma\mathbf{R}\times\mathbf{w}\,e^{i\gamma t}\\ &\quad+\alpha\sum_m\left[(A_{m0}+\alpha A_{m1}+\ldots)\left(1+\alpha d_1\frac{\partial}{\partial d_0}+\ldots\right)\mathbf{Q}_n+\alpha\mathbf{Q}_0+\ldots\right]\\ &\quad\times\left[\nabla\times\sum_m(A_{n0}+\alpha A_{n1}+\ldots)\left(1+\alpha d_1\frac{\partial}{\partial d_0}+\ldots\right)\mathbf{Q}_n+\alpha\mathbf{Q}_0+\ldots\right].\end{aligned} \tag{9}$$

At zeroth order in α we have

$$\sum_m (\partial A_{m0}/\partial t - i\lambda_{m0} A_{m0})\, \mathbf{Q}_m = i\gamma \mathbf{R} \times \mathbf{w}\, e^{i\gamma t}. \tag{10}$$

Now, using the orthogonality properties of the inertial modes, we get

$$A_{m0} = a_{m0}(\tau, T, \ldots) \exp(i\lambda_{m0} t) + i\gamma e^{i\gamma t} \int \mathbf{R} \times \mathbf{w} \,.\, \mathbf{Q}_m^* \, dV \Big/ (i\gamma - i\lambda_{m0}) \int \mathbf{Q}_m \,.\, \mathbf{Q}_m^* \, \partial V.$$

At first order in α, (9) gives

$$\sum_m \left\{ \left(\frac{\partial A_{m0}}{\partial t} - i\lambda m_0 A_{m0} \right) d_1 \frac{\partial}{\partial d_0} + \frac{\partial A_{m0}}{\partial \tau} + \frac{\partial A_{m1}}{\partial t} - i\lambda_{m0} A_{m1} \right\} \mathbf{Q}_m + 2\mathbf{k} \times \mathbf{Q}_0$$
$$= -\nabla P_0 + \sum_m A_{m0} \mathbf{Q}_m \times (\nabla \times \sum_n A_{n0} \mathbf{Q}_n). \tag{11}$$

From (10) and the orthogonality properties of the inertial modes, we have

$$\frac{\partial A_{m0}}{\partial \tau} + \frac{\partial A_{m1}}{\partial t} - i\lambda_{m0} A_{m1} + \left[\int \mathbf{Q}_m^* \,.\, (2\mathbf{k} \times \mathbf{Q}_0 + \nabla P_0)\, \partial V \right.$$
$$\left. - \Sigma\Sigma A_{l0} A_{n0} \int \mathbf{Q}_m^* \,.\, \mathbf{Q}_l \times \nabla \times \mathbf{Q}_n \, dV \right] \Big/ \int \mathbf{Q}_m \,.\, \mathbf{Q}_m^* \, dV = 0. \tag{12}$$

We take the fourth term equal to zero (we shall show later that this allows for a kind of geostrophy of the flow) by assuming that $\mathbf{Q}_0$ satisfies

$$2\mathbf{k} \times \mathbf{Q}_0 = -\nabla P_0. \tag{13}$$

If we note that $A_{l0} A_{n0}$ has an $\exp[i(\lambda_{l0} + \lambda_{n0})\, t]$ time dependence, we need not be concerned about uniform validity here and can take

$$A_{ml} = a_{m1}(\tau, T, \ldots) \exp(i\lambda_{m0} t), \quad \partial A_{m0}/\partial \tau = 0. \tag{14), (15}$$

At second order in α, (9) yields

$$\sum_m \left[\frac{\partial A_{m0}}{\partial T} + \frac{\partial A_{m1}}{\partial \tau} + \frac{\partial A_{m2}}{\partial t} - i\lambda_{m0} A_{m2} + \left(\frac{\partial A_{m1}}{\partial t} - i\lambda_{m0} A_{m1} \right) d_1 \frac{\partial}{\partial d_0} \right] \mathbf{Q}_{m0}$$
$$+ \frac{\partial \mathbf{Q}_0}{\partial \tau} + 2\mathbf{k} \times \mathbf{Q}_1 = -\nabla P_1 + \sum_m A_{m0} \mathbf{Q}_m \times (\nabla \times \mathbf{Q}_0) + \mathbf{Q}_0 \times (\nabla \times \sum_n A_{n0} \mathbf{Q}_{n0})$$
$$+ \sum_m A_{m0} \mathbf{Q}_{m0} \times (\nabla \times \sum_n A_{n1} \mathbf{Q}_{n0}) + \sum_m A_{m1} \mathbf{Q}_{m0} \times (\nabla \times \sum_n A_{n0} \mathbf{Q}_{n0})$$
$$+ \sum_m A_{m0} d_1 \frac{\partial \mathbf{Q}_{m0}}{\partial d_0} \times (\nabla \times \sum_n A_{n0} \mathbf{Q}_{n0}) + \sum_m A_{m0} \mathbf{Q}_{m0} \times \left(\nabla \times \sum_n A_{n0} d_1 \frac{\partial \mathbf{Q}_{n0}}{\partial d_0} \right). \tag{16}$$

Consistent with (13) we take

$$\partial \mathbf{Q}_0/\partial \tau + 2\mathbf{k} \times \mathbf{Q}_1 = -\nabla P_1. \tag{17}$$

Using (14) and again invoking orthogonality, we get

$$\frac{\partial A_{m0}}{\partial T} + \frac{\partial A_{m2}}{\partial t} - i\lambda_{m0} A_{m2} + \frac{\partial A_{m1}}{\partial \tau} = \int [\sum_l A_{l0} \mathbf{Q}_m^* \,.\, \mathbf{Q}_l \times (\nabla \times \mathbf{Q}_0)$$
$$+ \sum_n A_{n0} \mathbf{Q}_m^* \,.\, \mathbf{Q}_0 \times (\nabla \times \mathbf{Q}_n) + \sum_{l,n} A_{l0} A_{n1} \mathbf{Q}_m^* \,.\, \mathbf{Q}_l \times (\nabla \times \mathbf{Q}_{n0})$$
$$+ \sum_{l,n} A_{\ 1} A_{n1} \mathbf{Q}_m^* \,.\, \mathbf{Q}_l \times (\nabla \times \mathbf{Q}_n)\, dV] \Big/ \int \mathbf{Q}_{m0} \,.\, \mathbf{Q}_{m0}^* \, dV. \tag{18}$$

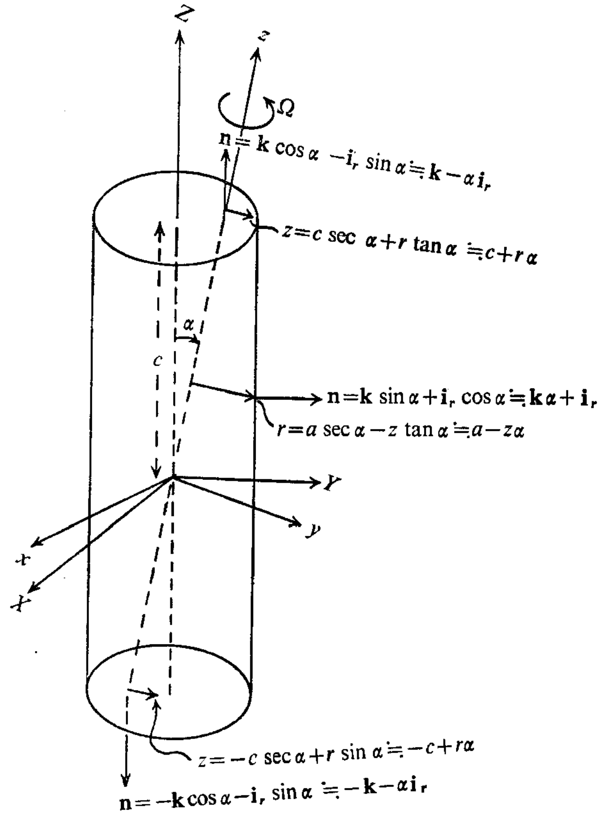

FIGURE 4. Geometrical conditions at the boundaries.

The contributions from $l = m$ and $n = m$ to the first two terms on the right-hand side of (18) make the solution not uniformly valid for large times. Hence we take

$$A_{m1} = a_{m1}(T, \psi, \ldots)\exp(i\lambda_{m0}t), \quad A_{m2} = a_{m2}(\tau, T, \ldots)\exp(i\lambda_{m0}t) \qquad (19), (20)$$

and

$$A_{m0} = a_{m0}(\psi, \ldots)\exp(i\lambda_{m0}t) + i\gamma e^{i\gamma t}\int \mathbf{R}\times\mathbf{w}\,.\,\mathbf{Q}^*_{m0}\,dV\,[(i\gamma - i\lambda_{m0})\int \mathbf{Q}_{m0}\cdot\mathbf{Q}_{m0}\,dV]^{-1}$$
$$+\exp\{\int \mathbf{Q}^*_{m0}\cdot[\mathbf{Q}_{m0}\times(\nabla\times\mathbf{Q}_0) + \mathbf{Q}_0\times(\nabla\times\mathbf{Q}_{m0})]\,T\,dV\}[\int \mathbf{Q}_{m0}\cdot\mathbf{Q}^*_{m0}\,dV]^{-1}. \qquad (21)$$

Equation (21) is the result we need to show the frequency-shift effect of the interaction between the inertial mode $\mathbf{Q}_{m0}$, given in (6), and the assumed Rossby mode $\mathbf{Q}_0$, for which we now find an explicit expression.

The vector product of (13) and $\mathbf{k}$ yields

$$\mathbf{k}\,.\,\mathbf{Q}_0 = 0, \quad \mathbf{Q}_0 = \tfrac{1}{2}\mathbf{k}\times\nabla P_0. \qquad (22), (23)$$

The scalar product of (13) and $\mathbf{k}$ yields

$$\partial P_0/\partial z = 0, \qquad (24)$$

which, from (23), implies that

$$\partial \mathbf{Q}_0/\partial z = 0. \qquad (25)$$

Hence the low frequency motion is geostrophic, i.e. the motion characterized by $\mathbf{Q}_0$ is columnar. To determine $\mathbf{Q}_0$ explictly we follow Pedlosky & Greenspan and derive a differential equation for P_0 by considering the boundary conditions on the ends (since P_0 is independent of z). In figure 4 we show the geometrical conditions on the ends (greatly magnified). We have at the top ($z = c$) and the bottom ($z = -c$), respectively,

$$\sum_m \Big\{(A_{m0}+\alpha A_{m1}+\dots)\Big(1+\alpha d_1\frac{\partial}{\partial d_0}+\dots\Big)\,\mathbf{Q}_{m0}+\alpha\mathbf{Q}_0+\alpha^2\mathbf{Q}_1+\dots$$
$$+r\alpha\frac{\partial}{\partial z}\Big[(A_{m0}+\alpha A_{m1}+\dots)\Big(1+\alpha d_1\frac{\partial}{\partial d_0}+\dots\Big)$$
$$\times\,\mathbf{Q}_{m0}+\alpha\mathbf{Q}_0+\alpha^2\mathbf{Q}_1+\dots\Big]\Big\}_{z=c}.(\mathbf{k}-\alpha\mathbf{i}_r) = 0 \tag{26}$$

and

$$\sum_m \Big\{(A_{m0}+\alpha A_{m1}+\dots)\Big(1+\alpha d_1\frac{\partial}{\partial d_0}+\dots\Big)\,\mathbf{Q}_{m0}+\alpha\mathbf{Q}_0+\alpha^2\mathbf{Q}_1+\dots$$
$$+r\alpha\frac{\partial}{\partial z}\Big[(A_{m0}+\alpha A_{m1}+\dots)\Big(1+\alpha d_1\frac{\partial}{\partial d_0}+\dots\Big)$$
$$\times\,\mathbf{Q}_{m0}+\alpha\mathbf{Q}_0+\alpha^2\mathbf{Q}_1+\dots\Big]\Big\}_{z=-c}.(-\mathbf{k}-\alpha\mathbf{i}_r) = 0. \tag{27}$$

The terms of zero order in α give (5). Hence, at first order in α,

$$\sum_m -A_{m0}\mathbf{Q}_{m0}\cdot\mathbf{i}_r+\mathbf{Q}_0\cdot\mathbf{k}-\sum_m A_{m1}\mathbf{Q}_{m0}\cdot\mathbf{k}+r\frac{\partial}{\partial z}\sum_m A_{m0}\mathbf{Q}_{m0}\cdot\mathbf{k}$$
$$+\sum_m A_{m0}d_1\frac{\partial}{\partial d_0}\mathbf{Q}_{m0}\cdot\mathbf{k} = 0. \tag{28}$$

Since $\mathbf{Q}_0.\mathbf{k} = 0$ from (22) and $\mathbf{Q}_{m0}.\mathbf{k} = 0$ from (5), we have

$$[\mathbf{Q}_{m0}\cdot\mathbf{i}_r-r\partial(\mathbf{Q}_{m0}\cdot\mathbf{k})/\partial z]_{z=c} = [d_1\,\partial(\mathbf{Q}_m\cdot\mathbf{k})/\partial d_0]_{z=c}. \tag{29}$$

At second order in α we have from (26)

$$\Big[-\sum_m A_{m1}\mathbf{Q}_{m0}\cdot\mathbf{i}_r-\mathbf{Q}_0\cdot\mathbf{i}_r+\mathbf{Q}_1\cdot\mathbf{k}-r\frac{\partial}{\partial z}\sum_m A_{m0}\mathbf{Q}_m\cdot\mathbf{i}_r$$
$$+r\frac{\partial}{\partial z}\sum_m A_{m1}\mathbf{Q}_m\cdot\mathbf{k}+r\frac{\partial}{\partial z}\mathbf{Q}_0\cdot\mathbf{k}\Big]_{z=c}$$
$$+\Big[-\Sigma A_{m0}d_1\frac{\partial}{\partial z}\mathbf{Q}_m\cdot\mathbf{i}_r+\Sigma A_{m0}r\frac{\partial}{\partial z}d_1\frac{\partial}{\partial d_0}\mathbf{Q}_m\cdot\mathbf{k}+\Sigma A_{m1}d_1\frac{\partial}{\partial d_0}\mathbf{Q}_m\cdot\mathbf{k}\Big]_{z=c} = 0. \tag{30}$$

From (29) we see that the first, fifth and ninth terms in this expression cancel. Also, from (6), $[\partial(\mathbf{Q}_m.\mathbf{i}_r)/\partial z]_{z=c} = 0$, and from (25), the sixth term vanishes. The seventh and eighth terms involve products of d_1, which we neglect. Hence

$$-\mathbf{Q}_0\cdot\mathbf{i}_r+[\mathbf{Q}_1\cdot\mathbf{k}]_{z=c} = 0. \tag{31}$$

Similarly, from (27), we have

$$\mathbf{Q}_0\cdot\mathbf{i}_r+[\mathbf{Q}_1\cdot\mathbf{k}]_{z=-c} = 0. \tag{32}$$

From (17) we determine $\mathbf{Q}_1$ by taking the curl, getting

$$\partial(\nabla\times\mathbf{Q}_0)/\partial\tau-2\,\partial\mathbf{Q}_1/\partial z = 0. \tag{33}$$

Since $\mathbf{Q}_0$ is independent of z, and since curl $\mathbf{Q}_0$ is in the direction of $\mathbf{k}$, a general enough integral of this equation is

$$\mathbf{Q}_1 = \tfrac{1}{2}z\,\partial(\nabla\times\mathbf{Q}_0)/\partial\tau + F_1(\mathbf{R},\tau,T,\ldots)\,\mathbf{i}_r + F_2(\mathbf{R},\tau,T,\ldots)\,\mathbf{i}_\theta, \tag{34}$$

where F_1 and F_2 are arbitrary functions. Substituting (34) into (31) and (32), one gets the same equation:

$$\partial(\nabla^2 P_0)/\partial\tau + (2/cr)\,\partial P_0/\partial\theta = 0, \tag{35}$$

which is the governing equation for the Rossby waves.

One of the boundary conditions for this equation follows from the restriction $\mathbf{q}\cdot\mathbf{n} = 0$ on the side wall:

$$\begin{aligned}\sum_m \Big\{ &(A_{m0}+\alpha A_{m1}+\ldots)\Big(1+\alpha d_1\frac{\partial}{\partial d_0}+\ldots\Big)\mathbf{Q}_{m0}+\alpha\mathbf{Q}_0+\alpha^2\mathbf{Q}_1+\ldots\\ &-z\alpha\frac{\partial}{\partial r}\Big[(A_{m0}+\alpha A_{m1}+\ldots)\Big(1+\alpha d_1\frac{\partial}{\partial d_0}+\ldots\Big)\mathbf{Q}_{m0}+\alpha\mathbf{Q}_0+\alpha^2\mathbf{Q}_1+\ldots\Big]\Big\}_{r=a}\\ &\times(\mathbf{i}_r+\alpha\mathbf{k}) = 0. \end{aligned}\tag{36}$$

At zero order in α we have

$$[\mathbf{Q}_{m0}\cdot\mathbf{i}_r]_{r=a} = 0, \tag{37}$$

which we already know from (5). Hence, at first order in α, we have

$$\sum_m(A_{m0}\mathbf{Q}_{m0}\cdot\mathbf{i}_r + A_{m0}\mathbf{Q}_{m0}\cdot\mathbf{k}) + \mathbf{Q}_0\cdot\mathbf{i}_r - z\frac{\partial}{\partial r}A_{m0}\Big[\mathbf{Q}_{m0}\cdot\mathbf{i}_r + A_{m0}d_1\frac{\partial}{\partial d_0}\mathbf{Q}_{m0}\cdot\mathbf{i}_r\Big]_{r=a} = 0. \tag{38}$$

Now $[\mathbf{Q}_{m0}\cdot\mathbf{i}_r] = 0$ from (37), and since $\mathbf{Q}_0$ is independent of z we must have

$$\Big[z\frac{\partial}{\partial r}\sum_m A_{m0}\mathbf{Q}_{m0}\cdot\mathbf{i}_r\Big]_{r=a} = \sum_m A_{m0}\Big[\mathbf{Q}_{m0}\cdot\mathbf{k} + d_1\frac{\partial}{\partial d_0}\mathbf{Q}_{m0}\cdot\mathbf{i}_r\Big]_{r=a} \tag{39}$$

and $$[\mathbf{Q}_0\cdot\mathbf{i}_r]_{r=a} = 0.$$

Hence $$P_0|_{r=a} = 0. \tag{40}$$

Similarly we can show that the boundary condition at the free surface is

$$P_0|_{r=b} = 0. \tag{41}$$

Hence, letting $P_0 = P_{00}e^{im\theta}e^{is\tau}$, the boundary-value problem for P_{00} is

$$\nabla^2 P_{00} + (2m/crs)\,P_{00} = 0, \tag{42}$$

where $$P_{00}(r=a) = P_{00}(r=b) = 0. \tag{43), (44}$$

Under the transformation $r = \zeta^2$ equation (42) becomes Bessel's equation of order $2m$, the solution of which in terms of r is

$$P_{00} = CJ_{2m}((8mr/sc)^{\frac{1}{2}}) + DY_{2m}((8mr/sc)^{\frac{1}{2}}), \tag{45}$$

where from (43) and (44)

$$CJ_{2m}((8ma/sc)^{\frac{1}{2}}) + DY_{2m}((8ma/sc)^{\frac{1}{2}}) = 0 \tag{46}$$

and

$$CJ_{2m}((8mb/sc)^{\frac{1}{2}}) + DY_{2m}((8mb/sc)^{\frac{1}{2}}) = 0. \tag{47}$$

Hence

$$D = -[J_{2m}((8ma/sc)^{\frac{1}{2}})/Y_{2m}((8mb/sc)^{\frac{1}{2}})]\,C \tag{48}$$

and the frequency equation is

$$J_{2m}((8ma/sc)^{\frac{1}{2}})\,Y_{2m}((8mb/sc)^{\frac{1}{2}}) - J_{2m}((8mb/sc)^{\frac{1}{2}})\,Y_{2m}((8ma/sa)^{\frac{1}{2}}) = 0. \tag{49}$$

In table 1 we give several representative values of s_1 (along with some corresponding values of λ taken from Stewartson's tables) for various combinations of c/a and b^2/a^2, where a is the cavity radius, $2c$ the cavity height and b the radius of the free surface of the liquid.

$m = 1$				$m = 2$		
c/a	b^2/a^2	s_1	λ	c/a	b^2/a^2	s_1
3	0	0·101	0·9957	3	0	0·093
3	0·16	0·033	0·976	3	0·16	0·017
3	0·25	0·022	0·933	3	0·25	0·00448
3	0·30	0·017	0·909			
3	0·50	0·006	0·792			

TABLE 1

$$P_0 = C\sum_m\sum_j\left\{J_{2m}\left(\left(\frac{8mr}{s_jc}\right)^{\frac{1}{2}}\right)\right.$$
$$\left.-\left[J_{2m}\left(\left(\frac{8ma}{s_jc}\right)^{\frac{1}{2}}\right)\Big/Y_{2m}\left(\left(\frac{8ma}{s_jc}\right)^{\frac{1}{2}}\right)\right]Y_{2m}\left(\left(\frac{8mr}{s_jc}\right)^{\frac{1}{2}}\right)\right\}\exp\left[i(lm\theta+s_j\tau)\right] \tag{50}$$

and thus

$$\mathbf{Q}_0 = \frac{C}{2}\sum_m\sum_j \exp\left[i(m\theta+s_j\tau)\right]\left(\mathbf{i}_\theta\frac{d}{dr}-\frac{im}{r}\mathbf{i}_r\right)\left\{J_{2m}\left(\left(\frac{8mr}{s_jc}\right)^{\frac{1}{2}}\right)\right.$$
$$\left.-J_{2m}\left(\left(\frac{8ma}{s_jc}\right)^{\frac{1}{2}}\right)\right\}\Big/Y_{2m}\left(\left(\frac{8ma}{s_jc}\right)^{\frac{1}{2}}\right)Y_{2m}\left(\left(\frac{8mr}{s_jc}\right)^{\frac{1}{2}}\right). \tag{51}$$

The interactions

We now use (6) and (51) in (21) with the hope that the integral in the exponential will be complex, thus providing the frequency shift indicated by figures 2 and 3. In that integral, we have

$$\mathbf{Q}^*_{m0}\cdot\mathbf{Q}_{m0}\times(\nabla\times\mathbf{Q}_0) = \mathbf{Q}^*_{m0}\cdot\mathbf{Q}_{m0}\times\mathbf{k}\nabla^2P_0, \tag{52}$$

which integrates to zero over the volume of liquid! Similarly

$$\mathbf{Q}^*_{m0}\cdot\mathbf{Q}_0\times(\nabla\times\mathbf{Q}_{m0}) = \tfrac{1}{2}\mathbf{Q}^*_{m0}\cdot(\mathbf{k}\times\nabla P_0)\times(\nabla\times\mathbf{Q}_{m0}) \tag{53}$$

integrates to zero over the volume of liquid. Hence

$$A_{m0} = a_{m0}(\psi,\ldots)\exp(i\lambda_m t) + i\gamma\, e^{i\gamma t}\int\mathbf{R}\times\mathbf{w}\,.\,\mathbf{Q}^*_m\,dV/(i\gamma-i\lambda_m)\int\mathbf{Q}_m\cdot\mathbf{Q}^*_m\,dV \tag{54}$$

and to second order in α the effect we are searching for via nonlinear interactions is not present.

Even if the integrals had not yielded zero, these interaction effects would have been of second order in α in time and hence would have been small. For that reason there is no need to carry the analysis to third order in α even if the integrals are then non-zero. The only hope for the Rossby-mode idea, it seems, is to allow the angle α to vary. On physical grounds, however, it is difficult to see how this variation in α could affect the wave motion to such an extent that the integrals would not yield zero.

The modified inertial modes: experimental verification

Having dismissed the Rossby-wave effect, we now consider the effect on the inertial modes of the non-zero value of α. Returning to (29), which involves the modification d_1 to d, we have

$$[\mathbf{Q}_{m0}\cdot\mathbf{i}_r - r\,\partial(\mathbf{Q}_{m0}\cdot\mathbf{k})/\partial z]_{z=c} = d_1[\partial(\mathbf{Q}_{m0}\cdot\mathbf{k})/\partial d_0]_{z=c}.$$

Since this must hold for all r, we let $r = a$, for then the first term vanishes and we have

$$d_1 = -a[\partial(\mathbf{Q}_{m0}\cdot\mathbf{k})/\partial z]_{z=c}[\partial(\mathbf{Q}_{m0}\cdot\mathbf{k})/\partial d_0]^{-1}. \tag{55}$$

Using (6) and (7) we can reduce (55) to

$$d_1 = -ad_0/2c, \tag{56}$$

where $d_0 = (2j+1)\pi/2c$. Hence the value of d to be used in determining the inertial wave frequency from (8) is

$$d = d_0 + \alpha d_1 = d_0(1 - a\alpha/2c) = (2j+1)\pi/2c \tag{57}$$

and the value of $c/a(2j+1)$ to use in Stewartson's tables is

$$[c/a(2j+1)] = [c/a(2j+1)]/(1 - a\alpha/2c). \tag{58}$$

Consider now figure 2, where we show two different growth-rate curves for a partially filled cavity. The growth-rate curve peaking at a gyroscope frequency of 0·0565 was generated by letting the amplitude of the gyroscopic motion increase from the sleeping position. The growth-rate curve peaking at a gyroscope frequency of 0·064 was generated by giving the gyroscopic motion an initial amplitude of about 3°. Let us determine whether (58) can predict the observed value of 0·064. From the figure, $a = 1{\cdot}25$ in. $2c = 7{\cdot}48$ in. and, for $j = 1$, $[c/a(2j+1)]_{\alpha=0} = 0{\cdot}997$. Hence, when $\alpha = 3° = 0{\cdot}052$ rad,

$$[c/a(2j+1)]_\alpha = 0{\cdot}997/(1 - a\alpha/2c) = 1{\cdot}006.$$

From Stewartson's tables, this gives an inertial wave frequency of 0·065, a value only slightly exceeding the experimental value of 0·064. Note that the shifts in the peaks do not seem to be viscosity dependent.

Consider now figure 3, where we show two different growth-rate curves for a completely filled cavity. Here the peak in the curve generated by letting the amplitude grow from the sleeping position occurs at a gyroscope frequency of 0·049. The growth-rate curve peaking at a gyroscope frequency of 0·053 was generated by giving the gyroscope an initial amplitude of about 3°. Here, $a = 1{\cdot}24$ in., $2c = 7{\cdot}817$ in. and $[c/a(2j+1)]_{\alpha=0} = 1{\cdot}0507$. As before, using

$\alpha = 0{\cdot}052\,\text{rad}$, we have $[c/a(2j+1)]_\alpha = 1{\cdot}0507/(1-0{\cdot}0082) = 1{\cdot}059$. Using Stewartson's tables, we get an inertial wave frequency of 0·054, again only slightly exceeding the experimental value.

3. Conclusions

The inertial wave eigenfrequencies of spinning liquids in cylinders depend very critically upon the geometry of the container. When the liquid is not spinning about the axis of the cylindrical cavity, it 'sees' a different geometry with a consequent change (given by a remarkably simple expression) in the wave frequency. The Rossby modes excited by the 'geometry' do not, to second order in the small amplitude of the gyroscope motion, have any effect on the inertial modes.

REFERENCES

Greenspan, H. P. 1969 *J. Fluid Mech.* **36**, 257.
Pedlosky, J. & Greenspan, H. P. 1967 *J. Fluid Mech.* **27**, 291.
Stewartson, K. 1959 *J. Fluid Mech.* **5**, 577.